If a region is conformally equivalent to the unit disc, then the region is simply connect proper subspace of the complex plane. Astonishingly, the converse holds as well.

We need two technical result at hands: Ascoli theorem (complex version) and injective sequence lemma. To describe our first result, we view region as a limit of compact subsets

$K_\ell \subset K_{\ell+1}$

$\{K\}_{\ell=1}^{\infty}$ is a nest sequence of compact subsets of Ω with $\Omega = \bigsqcup_{\ell=1}^{\infty} K_\ell$

say $\{K\}_{\ell=1}^{\infty}$ is an exhausting of Ω

As a consequence, we can describe the behavior of a family of functions on an exhausted region by its behavior on all compact subsets

$\mathcal{F}$ uniformly bounded $\Rightarrow$ equicontinuous

$$|f(z) - f(w)| = \left|\frac{1}{2\pi i}\int_\gamma f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) d\zeta\right| \leq \frac{B}{r}|z-w|$$

uniformly bounded + equicontinuous $\Rightarrow$ normal

Choose $\{w_i\}_{i=1}^{\infty}$ dense in Ω let $\{f_{n,0}\} \subset \mathcal{F}$

Since K compact, $\exists$ subsequence $\{f_{n,1}\}$ s.t. $f_{n,1}(w_1)$ converges

Choose $\{f_{n,2}\} \subset \{f_{n,1}\}$ s.t. $f_{n,2}(w_2)$ converges Repeat this

let $g_n = f_{n,n}$ then $g_n(w_j)$ converges for each $j \leq n$

By equicontinuity of $\mathcal{F}$, $\forall \varepsilon \; \exists \delta$ s.t. when $|z - w| < \delta$

$|g_n(z) - g_n(w)| < \varepsilon$ for all n. Since K compact, it's totally

bounded i.e. $K \subset \bigsqcup_{k=1}^{M} D_\delta(w_{i_k})$ Choose N large enough

s.t. $|g_m(w_{i_k}) - g_n(w_{i_k})| < \varepsilon$ for all $k = 1, \dots, N$ Hence given

$z \in K$ then $z \in D_\delta(w_{i_k})$ for some k.

$$\begin{tikzcd} g_n(z) \arrow[r, dash, "3\varepsilon"] \arrow[d, dash, "\varepsilon"'] & g_m(z) \arrow[d, dash, "\varepsilon"] \\ g_n(w_{i_k}) \arrow[r, dash, "\varepsilon"'] & g_n(w_{i_k}) \end{tikzcd}$$

Thus, $\{g_n\}$ uniformly converges on K

stationary

Now we spread this result onto the whole region:

information

Say $\{g_{n,1}\} \subset \{f_n\}$ converges uniformly on K_1

extract $\{g_{n,2}\} \subset \{g_{n,1}\}$ converges uniformly on K_2 Repeat this

Then $\{g_{n,n}\}$ converges uniformly on every K_l hence on Ω

If a sequence of injective holomorphic functions converges uniformly to a holomorphic function then it is either injective or constant.

Suppose for $z_1 \neq z_2$ in Ω, $f(z_1) = f(z_2)$ Estimate $g_n(z) = f_n(z) - f_n(z_1)$

g_n only roots in z_1 and $g_n \rightrightarrows g(z) = f(z) - f(z_1)$ Say g is not constantly zero then z_2 is an isolated zero

then $1 = \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)} d\zeta$ Notice $\frac{g_n'}{g_n} \rightrightarrows \frac{g'}{g}$

hence $\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)} d\zeta \to \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)} d\zeta$ but the former is constantly zero

Notice that we can deform arbitrary region to locate at the center of the plane.

$\Omega \xrightarrow{\log} \xrightarrow{\mathrm{pow}(-1)}$ bounded at center

idea: $f(w) + 2\pi i$ will leave $\mathrm{im} f$

We shall pick the desired functions whose derivative at origin takes the supremum thus automatically becomes conformal equivalence from a bunch candidates.

$\mathcal{F} = \{ f : \Omega \to \mathbb{D}$ holomorphic , injective and $f(0) = 0 \}$

$id \in \mathcal{F}$ nonempty $\qquad$ $\mathcal{F}$ uniformly bounded

notice $|f'(0)| = \left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta^2} \right|$ uniformly bounded

$s \geq 1$, as $id \in \mathcal{F}$

Choose $\{f_n\} \subset \mathcal{F}$ s.t. $|f_n'(0)| \to s = \sup_{f \in \mathcal{F}} |f'(0)|$ as $n \to +\infty$

There's subsequence converges uniformly on compact sets to

holomorphic f on Ω by Montel's Thm. $\quad$ Being nonconstant,

f has to be injective $\quad$ Since $\begin{cases} f_n(0) = 0 \\ |f_n| < 1 \end{cases}$, $\begin{cases} f(0) = 0 \\ |f| \leq 1 \end{cases}$. Moreover,

$|f(z)| < 1$ by maximum modules principle $\quad$ Therefore $f \in \mathcal{F}$ with $|f'(0)| = s$

The maximal element will automatically becomes surjective.

Suppose $\exists\, \alpha \in \mathbb{D} \setminus \operatorname{im} f$ $\qquad$ Consider Cayley map Ψ_α

$$\begin{tikzcd}
\Omega \arrow[r, "f"] & {} \arrow[r, dashed, "F"] \arrow[d, "\Psi_\alpha"'] & {} \arrow[r, dashed, "\Phi"] \arrow[d, shift left, "\Psi^{-1}_{g(\alpha)}"] & {} \\
 & U \arrow[r, "g"', "\mathrm{pow}(\frac{1}{2})"'] & {} \arrow[u, shift left, "\Psi_{g(\alpha)}"] \arrow[r, "h"', "\mathrm{pow}(2)"'] & {} \arrow[u, "\Psi^{-1}_\alpha"']
\end{tikzcd}$$

Φ is not injective since h is not

forces $|\Phi'(0)| < 1$ failing to be rotation

$f'(0) = \Phi'(0) F'(0)$ implies $|f'(0)| < |F'(0)|$